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On renormalisation of the quantum stress tensor in curved space–time by dimensional regularisation

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Abstract. Using dimensional regularisation, a prescription is given for obtaining a finite renormalised stress tensor in curved space–time. Renormalisation is carried out by renormalising coupling constants in the n-dimensional Einstein equation generalised to include tensors which are fourth order in derivatives of the metric. Except for the special case of a massless conformal field in a conformally flat space–time, this procedure is not unique. There exists an infinite one-parameter family of renormalisation *ansätze* differing from each other in the finite renormalisation that takes place. Nevertheless, the renormalised stress tensor for a conformally invariant field theory acquires a nonzero trace which is independent of the renormalisation *ansatz* used and which has a value in agreement with that obtained by other methods. A comparison is made with some earlier work using dimensional regularisation which is shown to be in error.

1. Introduction

One of the difficulties encountered in the problem of constructing a well defined quantum theory of matter fields propagating in an unquantised background space–time is the appearance of infinities in matrix elements of the quantum stress tensor, $T_{\mu\nu}$. In particular, the expectation value of $T_{\mu\nu}$ in some state of the quantum field, which will be written $\langle 0|T_{\mu\nu}|0\rangle$ since in practice the state is usually taken to be the vacuum, is infinite and must be renormalised if it is to be used as the source in Einstein's equations. In recent years a number of different regularisation schemes have been employed to deal with these divergences (Zel'dovich and Starobinsky 1971, 1972, Parker and Fulling 1974, Candelas and Raine 1975, Dowker and Critchley 1976, Davies *et al* 1976, Christensen 1976, Brown 1976, Brown and Cassidy 1976, Hawking 1977, Bernard and Duncan 1977) and in a number of cases explicit finite expressions have been obtained for the renormalised stress tensor $\langle 0|T_{\mu\nu}|0\rangle_{\text{ren}}$ in particular space–times (Candelas and Raine 1975, Dowker and Critchley 1976, Davies *et al* 1976, Bernard and Duncan 1977, Davies *et al* 1977, Bunch and Davies 1978a, b, Brown and Cassidy 1977). The most striking result which has been obtained by these methods is the appearance of a nonzero trace in $\langle 0|T_{\mu\nu}|0\rangle_{\text{ren}}$ for fields which are conformally invariant and which therefore have classical stress tensors whose traces vanish identically (Dowker and Critchley 1976, Davies *et al* 1976, Hawking 1977, Deser *et al* 1976, Christensen and Fulling 1977, Duff 1977, Tsao 1977). In order to understand how this trace anomaly arises, it is important to make a distinction between regularisation and renormalisation. Regularisation involves redefining $\langle 0|T_{\mu\nu}|0\rangle$ in such a way that it becomes dependent on some parameter, or parameters, ϵ. As long as $\epsilon \neq 0$, $\langle 0|T_{\mu\nu}(\epsilon)|0\rangle$ is finite: the infinite quantity $\langle 0|T_{\mu\nu}|0\rangle$ is recovered by removing the regularisation—that is, by letting $\epsilon \to 0$. Renormalisation involves separating $\langle 0|T_{\mu\nu}(\epsilon)|0\rangle$ into two parts, one of which is divergent as $\epsilon \to 0$ and the other finite:

$$\langle 0|T_{\mu\nu}(\epsilon)|0\rangle = \langle 0|T_{\mu\nu}(\epsilon)|0\rangle_{\rm div} + \langle 0|T_{\mu\nu}(\epsilon)|0\rangle_{\rm fin} \tag{1.1}$$

where

$$\lim_{\epsilon\to 0} \langle 0|T_{\mu\nu}(\epsilon)|0\rangle_{\rm div} = \infty \qquad \lim_{\epsilon\to 0} \langle 0|T_{\mu\nu}(\epsilon)|0\rangle_{\rm fin} < \infty.$$

The renormalisation is completed by discarding $\langle 0|T_{\mu\nu}(\epsilon)|0\rangle_{\rm div}$ and letting $\epsilon \to 0$ to obtain

$$\langle 0|T_{\mu\nu}|0\rangle_{\rm ren} = \lim_{\epsilon\to 0} \langle 0|T_{\mu\nu}(\epsilon)|0\rangle_{\rm fin} = \lim_{\epsilon\to 0} [\langle 0|T_{\mu\nu}(\epsilon)|0\rangle - \langle 0|T_{\mu\nu}(\epsilon)|0\rangle_{\rm div}]. \tag{1.2}$$

Provided that the regularisation scheme respects all the invariances of the theory including conformal invariance, $\langle 0|T_{\mu\nu}(\epsilon)|0\rangle$ will be traceless if $T_{\mu\nu}$ is. Consequently

$$g^{\mu\nu}\langle 0|T_{\mu\nu}|0\rangle_{\rm ren} = -\lim_{\epsilon\to 0} g^{\mu\nu}\langle 0|T_{\mu\nu}(\epsilon)|0\rangle_{\rm div}. \tag{1.3}$$

Thus if the renormalised stress tensor is to acquire an anomalous trace, the divergent quantity $\langle 0|T_{\mu\nu}(\epsilon)|0\rangle_{\rm div}$ must have a finite nonzero trace in the limit $\epsilon \to 0$ equal to the negative of the anomalous trace.

The main difficulties with renormalisation are how to make the separation (1.1) and how to justify discarding the divergent terms. One expects that these problems can be tackled by remembering that $\langle 0|T_{\mu\nu}|0\rangle$ is the source term in Einstein's equation, which in four dimensions is:

$$\Lambda g_{\mu\nu} + G_{\mu\nu} + \lambda_1 {}^{(1)}H_{\mu\nu} + \lambda_2 {}^{(2)}H_{\mu\nu} = 8\pi G\langle 0|T_{\mu\nu}|0\rangle \tag{1.4}$$

where Λ is the cosmological constant, G is Newton's constant and $G_{\mu\nu}$ Einstein's tensor. The two conserved tensors ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$ are defined by

$${}^{(1)}H_{\mu\nu} \equiv \frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}\int \sqrt{g}R^2\,{\rm d}^4x = 2R_{;\mu\nu} - 2\Box R g_{\mu\nu} + \tfrac{1}{2}R^2 g_{\mu\nu} - 2RR_{\mu\nu} \tag{1.5}$$

$$\begin{aligned}{}^{(2)}H_{\mu\nu} &\equiv \frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}\int \sqrt{g}R^{\alpha\beta}R_{\alpha\beta}\,{\rm d}^4x \\ &= \tfrac{1}{2}R^{\alpha\beta}R_{\alpha\beta}g_{\mu\nu} - 2R^{\alpha\beta}R_{\alpha\mu\beta\nu} + R_{;\mu\nu} - \Box R_{\mu\nu} - \tfrac{1}{2}\Box R g_{\mu\nu}.\end{aligned} \tag{1.6}$$

Note that in classical relativity one always has $\lambda_1 = \lambda_2 = 0$ and usually $\Lambda = 0$. The tensors $g_{\mu\nu}$, ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$ are required in the present theory to assist in the renormalisation of $\langle 0|T_{\mu\nu}|0\rangle$. However, after renormalisation, the renormalised coefficients of these tensors may presumably be taken to be zero. Because of the Gauss–Bonnet theorem in four dimensions, which implies that

$$\frac{\delta}{\delta g^{\mu\nu}}\int \sqrt{g}\,(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 4R^{\alpha\beta}R_{\alpha\beta} + R^2)\,{\rm d}^4x = 0 \tag{1.7}$$

the tensor obtained by varying $\int \sqrt{g}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}\,{\rm d}^4x$ with respect to the metric is a linear combination of ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$. The separation (1.1) is now made by taking $\langle 0|T_{\mu\nu}(\epsilon)|0\rangle_{\rm div}$ to be a linear combination of tensors appearing on the left-hand side of equation (1.4), with coefficients which diverge as $\epsilon \to 0$. $\langle 0|T_{\mu\nu}(\epsilon)|0\rangle_{\text{div}}$ is then transferred to the left-hand side of (1.4) and the new coefficients thus obtained are taken to be the observable coefficients whose finite values are to be determined by experiment. Putting $\epsilon = 0$ leaves $\langle 0|T_{\mu\nu}|0\rangle_{\text{ren}}$ on the right-hand side of equation (1.4) and the theory is now free of divergences. The only problem with this approach is that the tensors ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$ both have traces proportional to $\Box R$, so that this appears to be the only quantity which can arise in the anomalous trace, the traces of $G_{\mu\nu}$ and $g_{\mu\nu}$ being of the wrong dimension to appear in $\langle 0|T^{\alpha}_{\alpha}|0\rangle_{\text{ren}}$ when the quantum field is massless. However, it is now well established (Deser *et al* 1976, Christensen and Fulling 1977, Duff 1977, Tsao 1977) that the anomalous trace contains terms proportional to $\Box R$, $R^{\alpha\beta}R_{\alpha\beta} - \frac{1}{3}R^2$ and $C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}$, where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor. Because of this difficulty, few attempts have been made to renormalise $\langle 0|T_{\mu\nu}|0\rangle$ by renormalising coupling constants in Einstein's equation. An early paper based on adiabatic regularisation (Fulling and Parker 1974) was able to remove divergences from $\langle 0|T_{\mu\nu}|0\rangle$ but ran into difficulties over some finite terms of precisely the kind associated with the trace anomaly. The finite terms appearing in equation (3.20) of Fulling and Parker (1974) are, in spite of appearances, entirely independent of the mass so that the inability to remove these terms by renormalisation occurs for both massive and massless field theories.

An alternative approach to the renormalisation problem, and one that has frequently been used in practice, is to look at the action which leads to equation (1.4). This action may be written

$$S = S_{\mathrm{G}} + W = \int \sqrt{g}\mathscr{L}_{\mathrm{G}}\, \mathrm{d}^4x + \int \sqrt{g}\mathscr{L}^{(1)}\, \mathrm{d}^4x \tag{1.8}$$

and the field equations (1.4) are given by:

$$\delta S/\delta g^{\mu\nu} = 0. \tag{1.9}$$

The gravitational Lagrangian $\mathscr{L}_{\mathrm{G}}$ is a linear combination of 1, R, R^2 and $R^{\alpha\beta}R_{\alpha\beta}$ so that the left-hand side of equation (1.4) is obtained from:

$$\frac{2}{\sqrt{g}}\frac{\delta S_{\mathrm{g}}}{\delta g^{\mu\nu}} = \frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}\int \sqrt{g}\mathscr{L}_{\mathrm{G}}\, \mathrm{d}^4x \tag{1.10}$$

where $\mathscr{L}^{(1)}(x)$ is the effective Lagrangian for the field theory and W is the effective action. $\mathscr{L}^{(1)}(x)$ is divergent and must be regularised to give

$$\mathscr{L}^{(1)}(x;\epsilon) = \mathscr{L}^{(1)}_{\text{div}}(x;\epsilon) + \mathscr{L}^{(1)}_{\text{fin}}(x;\epsilon). \tag{1.11}$$

$\mathscr{L}^{(1)}(x)$ is now renormalised by absorbing $\mathscr{L}^{(1)}_{\text{div}}(x;\epsilon)$ into $\mathscr{L}_{\mathrm{G}}$ with renormalisation of coupling constants, leaving $\mathscr{L}^{(1)}_{\text{fin}}(x;\epsilon)$. The connection between this approach and the previous one is that

$$\langle 0|T_{\mu\nu}|0\rangle_{\text{div}} = \frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}\int \sqrt{g}\mathscr{L}^{(1)}_{\text{div}}\, \mathrm{d}^4x. \tag{1.12}$$

Renormalisation of the action is an apparent improvement over renormalisation of the stress tensor since the inconsistencies between the traces of $\langle 0|T_{\mu\nu}|0\rangle_{\text{ren}}$ and $\langle 0|T_{\mu\nu}(\epsilon)|0\rangle_{\text{div}}$ do not arise. Indeed, this approach has been used before and forms the basis of many of the derivations of the anomaly (Deser *et al* 1976, Duff 1977, Tsao

1977). Nevertheless it is disturbing that two very closely connected procedures appear to be inconsistent with one another.

The main purpose of this paper is to show how this inconsistency can be removed, and a well defined renormalisation prescription given which can be implemented either in the action for the theory or in the field equations (1.4). The method of regularisation which will be used is dimensional regularisation and the essential idea is the recognition that renormalisation of coupling constants must take place before the regularisation is removed. This means that both the effective action and the stress tensor must be renormalised in n dimensions. Dimensional regularisation has been used before in an attempt to define a renormalised stress tensor in curved space–time (Brown 1976, Brown and Cassidy 1976). However, a number of mistakes were made in that work and the secondary purpose of this paper is to examine Brown (1976) in detail in order to clarify how to perform the correct renormalisation of the quantum stress tensor by dimensional regularisation. For simplicity the quantum field is taken to be a scalar field; fields of higher spin could be treated in exactly the same way and will not therefore be discussed. Sign conventions are the same as those used in Christensen (1976) and Brown (1976).

2. Renormalisation of the quantum stress tensor

The action functional for a scalar field $\phi(x)$ in an n-dimensional space–time with metric $g_{\mu\nu}$ is:

$$S[\phi] = -\tfrac{1}{2}\int \sqrt{g}(g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi + \xi R\phi^2 + m^2\phi^2)\,\mathrm{d}^n x \tag{2.1}$$

where $g = -\det g_{\mu\nu}$, m is the mass of the scalar field, R the Ricci scalar and ξ a constant. If the theory based on $S[\phi]$ is to be conformally invariant in n dimensions when $m = 0$, one requires

$$\xi = (n-2)/4(n-1). \tag{2.2}$$

The scalar field equation, obtained by varying S with respect to ϕ, is:

$$\Box\phi - (\xi R + m^2)\phi = 0 \tag{2.3}$$

where $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$. The stress tensor is

$$\begin{aligned} T_{\mu\nu} &= \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}} \\ &= (1-2\xi)\,\partial_\mu\phi\,\partial_\nu\phi + (2\xi - \tfrac{1}{2})g_{\mu\nu}\,\partial^\alpha\phi\,\partial_\alpha\phi - 2\xi\phi\nabla_\mu\,\partial_\nu\phi \\ &\quad + 2\xi g_{\mu\nu}\phi\nabla^\alpha\,\partial_\alpha\phi + \xi G_{\mu\nu}\phi^2 - \tfrac{1}{2}m^2 g_{\mu\nu}\phi^2. \end{aligned} \tag{2.4}$$

Although the quantity which acts as the source in Einstein's equation (1.4) is the expectation value of $T_{\mu\nu}$ in some quantum state, De Witt (1975) has shown that the divergences in this quantity are identical to the divergences in

$$\langle T_{\mu\nu}\rangle \equiv \frac{\langle \mathrm{out}|T_{\mu\nu}|\mathrm{in}\rangle}{\langle \mathrm{out}|\mathrm{in}\rangle} \tag{2.5}$$

where $|\mathrm{in}\rangle$ and $|\mathrm{out}\rangle$ are vacuum states in initial and final static regions of the space–time which, for the purposes of obtaining the divergences in (2.5) and hence in $\langle 0|T_{\mu\nu}|0\rangle$, are assumed to exist although they need not exist in the particular space–time in which the state $|0\rangle$ is defined. For a space–time which contains such static regions but which is otherwise arbitrary, $\langle T_{\mu\nu}\rangle$ can be obtained from the effective action which is defined to be

$$W = -\mathrm{i}\log\langle\mathrm{out}|\mathrm{in}\rangle. \tag{2.6}$$

Under variations with respect to $g_{\mu\nu}$:

$$\delta W = -\mathrm{i}\frac{\delta\langle\mathrm{out}|\mathrm{in}\rangle}{\langle\mathrm{out}|\mathrm{in}\rangle}. \tag{2.7}$$

Schwinger's variational principle (Schwinger 1951) states that

$$\delta\langle\mathrm{out}|\mathrm{in}\rangle = \mathrm{i}\langle\mathrm{out}|\delta S|\mathrm{in}\rangle \tag{2.8}$$

and hence

$$\frac{2}{\sqrt{g}}\frac{\delta W}{\delta g^{\mu\nu}} = \frac{\langle\mathrm{out}|T_{\mu\nu}|\mathrm{in}\rangle}{\langle\mathrm{out}|\mathrm{in}\rangle} \equiv \langle T_{\mu\nu}\rangle. \tag{2.9}$$

In n dimensions, the effective action W may be written

$$W = \int \mathscr{L}^{(1)}(x)\sqrt{g}\,\mathrm{d}^n x \tag{2.10}$$

where $\mathscr{L}^{(1)}(x)$ is the effective Lagrangian which has a representation of the form

$$\mathscr{L}^{(1)}(x) = \frac{1}{2(4\pi)^{n/2}}\int_0^\infty \frac{\mathrm{i}\,\mathrm{d}s}{(\mathrm{i}s)^{1+n/2}}\exp(-\mathrm{i}m^2 s)F(x, \mathrm{i}s). \tag{2.11}$$

This representation for the effective action is well known (Dowker and Critchley 1976, Brown 1976, De Witt 1975) and details of its derivation will not be given here. The function $F(x, \mathrm{i}s)$ satisfies the boundary condition

$$F(x, 0) = 1. \tag{2.12}$$

Convergence of the integral in equation (2.11) at the upper limit of integration is guaranteed by the boundary conditions on the Feynman propagator from which $\mathscr{L}^{(1)}(x)$ is obtained, which require that m^2 contain a small negative imaginary part. Consequently the divergences all occur at the lower limit of integration at which $s = 0$, and it therefore suffices to expand $F(x, \mathrm{i}s)$ in a power series about $s = 0$:

$$F(x, \mathrm{i}s) = \sum_{k=0}^{\infty} a_k(x)(\mathrm{i}s)^k. \tag{2.13}$$

The coefficients $a_k(x)$ in this expansion are all geometric scalars and, on dimensional grounds, each term in $a_k(x)$ must contain $2k$ derivatives of the metric. The first three of these have been calculated elsewhere (Christensen 1976, Brown 1976, Brown and Cassidy 1976); they are

$$a_0(x) = 1 \tag{2.14}$$

$$a_1(x) = (\tfrac{1}{6} - \xi)R \tag{2.15}$$

$$a_2(x) = \tfrac{1}{180}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - \tfrac{1}{180}R^{\alpha\beta}R_{\alpha\beta} + \tfrac{1}{6}(\tfrac{1}{5} - \xi)\Box R + \tfrac{1}{2}(\tfrac{1}{6} - \xi)^2 R^2. \tag{2.16}$$

Substituting equation (2.13) in (2.11) and integrating,

$$\mathscr{L}^{(1)}(x)=\frac{1}{2(4\pi)^{n/2}}\sum_{k=0}^{\infty} a_k(x)m^{n-2k}\Gamma(k-n/2). \tag{2.17}$$

From this expression, it is easy to see that if $n=4$, the Γ function has poles for $k=0$, 1 and 2. These poles will eventually be removed by renormalising coupling constants in $\mathscr{L}_G$. In order to do this consistently it is necessary that both $\mathscr{L}_G$ and $\mathscr{L}^{(1)}$ have the same dimensions, and the most convenient way of ensuring this is to keep the dimensions of $\mathscr{L}^{(1)}$ fixed for arbitrary space–time dimension n by introducing an arbitrary parameter κ having dimensions of mass, and to rewrite (2.17) as

$$\mathscr{L}^{(1)}(x)=\frac{1}{2(4\pi)^{n/2}}(m/\kappa)^{n-4}\sum_{k=0}^{\infty} a_k(x)m^{4-2k}\Gamma(k-n/2). \tag{2.18}$$

Although it is not strictly necessary to what follows, it is convenient to make the following expansions in equation (2.18):

$$(m/\kappa)^{n-4}=1+\tfrac{1}{2}(n-4)\ln(m^2/\kappa^2)+\mathrm{O}(n-4)^2 \tag{2.19}$$

$$\Gamma(2-n/2)=\frac{2}{4-n}-\gamma+\mathrm{O}(n-4) \tag{2.20}$$

$$\Gamma\left(1-\frac{n}{2}\right)=\frac{2}{2-n}\left(\frac{2}{4-n}-\gamma\right)+\mathrm{O}(n-4) \tag{2.21}$$

$$\Gamma\left(-\frac{n}{2}\right)=\frac{4}{n(n-2)}\left(\frac{2}{4-n}-\gamma\right)+\mathrm{O}(n-4). \tag{2.22}$$

Then equation (2.18) becomes

$$\mathscr{L}^{(1)}(x)=-\frac{1}{(4\pi)^{n/2}}\left[\frac{1}{n-4}+\frac{1}{2}\left(\gamma+\ln\frac{m^2}{\kappa^2}\right)\right]\left[\frac{4m^4a_0}{n(n-2)}-\frac{2m^2a_1}{n-2}+a_2\right]$$
$$+\frac{1}{2(4\pi)^{n/2}}\sum_{k=3}^{\infty} a_k(x)m^{4-2k}\Gamma\left(k-\frac{n}{2}\right) \tag{2.23}$$

where terms of order $(n-4)$, which disappear when the limit $n\to 4$ is taken after the renormalisation has been performed, have been omitted. Expression (2.23) is the dimensionally regularised effective Lagrangian, the regularisation parameter being $\epsilon=n-4$. Because of the structure of a_0, a_1 and a_2 (equations (2.14)–(2.16)) it is possible to remove the divergent pole terms provided that the gravitational Lagrangian contains terms which are fourth order in derivatives of the metric. There is nevertheless some ambiguity in precisely which terms in (2.23) should be removed by renormalisation and the following two renormalisation *ansatze* suggest themselves:

(i) Remove only the pole terms; this means that the divergent part of the effective Lagrangian is taken to be

$$\mathscr{L}^{(1)}_{\text{div}}(x)=-\frac{1}{(4\pi)^{n/2}(n-4)}\left[\frac{4m^2a_0}{n(n-2)}-\frac{2m^2a_1}{n-2}+a_2\right]. \tag{2.24}$$

(ii) Remove all terms which are fourth order or lower in derivatives of the metric: the divergent part of the effective Lagrangian is then

$$\mathscr{L}^{(1)}_{\rm div}(x) = -\frac{1}{(4\pi)^{n/2}}\left[\frac{1}{n-4}+\frac{1}{2}\left(\gamma+\ln\frac{m^2}{\kappa^2}\right)\right]\left[\frac{4m^4a_0}{n(n-2)}-\frac{2m^2a_1}{n-2}+a_2\right]. \qquad (2.25)$$

Ansatz (ii) is really an infinite one-parameter family of *ansatze* since the parameter κ is arbitrary. *Ansatz* (i) is a special case of (ii) obtained by taking

$$\kappa = me^{\gamma/2}. \qquad (2.26)$$

Consequently, in what follows expression (2.25) will be used for $\mathscr{L}^{(1)}_{\rm div}(x)$ and a corresponding expression for $\langle T_{\mu\nu}\rangle_{\rm div}$ will be obtained by functional differentiation with respect to $g_{\mu\nu}$. This expression will contain a finite term proportional to $\gamma+\ln(m^2/\kappa^2)$ which would not have appeared if equation (2.24) has been taken as the starting point. In order to calculate $\langle T_{\mu\nu}\rangle_{\rm div}$, the following expressions valid in n dimensions will be required:

$$H_{\mu\nu} \equiv \frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}\int \sqrt{g}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}\,{\rm d}^n x$$

$$= \tfrac{1}{2}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}g_{\mu\nu} - 2R_{\mu\alpha\beta\gamma}R_{\nu}^{\ \alpha\beta\gamma} - 4\Box R_{\mu\nu} + 2R_{;\mu\nu} + 4R_{\mu\alpha}R^{\alpha}_{\ \nu} - 4R^{\alpha\beta}R_{\alpha\mu\beta\nu} \qquad (2.27)$$

$$\frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}\int \sqrt{g}R\,{\rm d}^n x = \frac{1}{2}g_{\mu\nu}R - R_{\mu\nu} = -G_{\mu\nu} \qquad (2.28)$$

$$\frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}\int \sqrt{g}\,{\rm d}^n x = \frac{1}{2}g_{\mu\nu}. \qquad (2.29)$$

Using equations (2.27)–(2.29) and also equations (1.5) and (1.6) which are valid in n dimensions, and substituting for a_0, a_1 and a_2 from equations (2.14)–(2.16), one finds

$$\langle T_{\mu\nu}\rangle_{\rm div} = -\frac{1}{(4\pi)^{n/2}}\left[\frac{1}{n-4}+\frac{1}{2}\left(\gamma+\ln\frac{m^2}{\kappa^2}\right)\right]$$
$$\times\left[\frac{4m^4g_{\mu\nu}}{n(n-2)}+\frac{4m^2(\frac{1}{6}-\xi)}{n-2}G_{\mu\nu}+\tfrac{1}{90}H_{\mu\nu}-\tfrac{1}{90}{}^{(2)}H_{\mu\nu}+(\tfrac{1}{6}-\xi)^2\,{}^{(1)}H_{\mu\nu}\right]. \qquad (2.30)$$

This expression for the divergences in $\langle T_{\mu\nu}\rangle$ is valid for arbitrary ξ and may be considered to be equivalent to expressions (6.2)–(6.6) of Christensen (1976) which were obtained by regularising $\langle T_{\mu\nu}\rangle$ using point-splitting. It is evident from equation (2.30) that $\langle 0|T_{\mu\nu}|0\rangle$ can be renormalised by renormalising coupling constants in the n dimensional Einstein equation which reads:

$$\Lambda g_{\mu\nu} + G_{\mu\nu} + \lambda H_{\mu\nu} + \lambda_1\,{}^{(1)}H_{\mu\nu} + \lambda_2\,{}^{(2)}H_{\mu\nu} = 8\pi G\langle 0|T_{\mu\nu}|0\rangle. \qquad (2.31)$$

Because the Gauss–Nonnet theorem (1.7) holds only in four dimensions, equation (2.13) contains one more tensor than (1.4). Unlike the attempt at renormalisation in four dimensions outlined in § 1, renormalisation in n dimensions is consistent with the existence of the trace anomaly—indeed, the trace anomaly is a natural consequence of this renormalisation. For $m=0$ and $\xi=(n-2)/4(n-1)$, the trace of equation (2.30) is finite in the limit $n\to 4$:

$$\lim_{n\to 4} g^{\mu\nu}\langle T_{\mu\nu}\rangle_{\rm div}$$

$$= -\frac{1}{2880\pi^2}(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - R^{\alpha\beta}R_{\alpha\beta} + \Box R) \tag{2.32}$$

$$= -\frac{1}{2880\pi^2}(C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} + R^{\alpha\beta}R_{\alpha\beta} - \tfrac{1}{3}R^2 + \Box R). \tag{2.33}$$

This is the negative of the trace anomaly (Christensen and Fulling 1977, Tsao 1977) as required by equation (1.3). The anomaly arises directly from the pole terms and does not depend on the term $\gamma+\ln(m^2/\kappa^2)$. Consequently, the anomaly is independent of which renormalisation *ansatz* is used.

It has been argued by Dowker and Critchley (1977) that it is necessary to discard the finite terms proportional to $\ln(m^2/\kappa^2)$ in order to acquire a trace anomaly. Their argument is that since the trace of $\langle T_{\mu\nu}\rangle$ is from equation (2.4) using the field equation (2.3):

$$\langle T^\alpha_\alpha\rangle = -m^2\langle\phi^2\rangle = \mathrm{i}m^2 G_{\rm F}(x,x') \tag{2.34}$$

where $G_{\rm F}(x,x')$ is the Feynman Green's function, and since

$$G_{\rm F}(x,x) = -2\mathrm{i}\,\partial\mathscr{L}^{(1)}/\partial m^2 \tag{2.35}$$

it follows that:

$$\langle T^\alpha_\alpha\rangle = 2m^2\frac{\partial\mathscr{L}^{(1)}}{\partial m^2}. \tag{2.36}$$

Thus, by equation (1.3)

$$\langle 0|T^\alpha_\alpha|0\rangle_{\rm ren} = -\lim_{m\to 0}\lim_{n\to 4} 2m^2\frac{\partial\mathscr{L}^{(1)}_{\rm div}}{\partial m^2} \tag{2.37}$$

and the result $\langle 0|T^\alpha_\alpha|0\rangle_{\rm ren} = a_2(x)/16\pi^2$ is obtained if and only if $\mathscr{L}^{(1)}_{\rm div}$ contains the term $a_2(x)\ln(m^2/\kappa^2)$. The flaw in this argument lies in equation (2.34). Although it is true that the mass-independent terms in equation (2.4) are traceless and hence that the dimensionally regularised expression arising from these terms is also traceless, the pole terms alone are not traceless. Discarding only the pole terms (*ansatz* (i)) gives rise to the anomalous trace, not from

$$\lim_{m\to 0} m^2\langle\phi^2\rangle$$

which contains no poles, but from the mass-independent terms in equation (2.4). The equivalence of the two routes to the trace anomaly, via either the pole terms or, using equation (2.37), the $\ln(m^2/\kappa^2)$ term, is a reflection of the fact that, for $\xi=\frac{1}{6}$:

$$\frac{2}{\sqrt{g^{\mu\nu}}}\frac{\delta}{\delta g^{\mu\nu}}\int\sqrt{g}a_2(x)\,\mathrm{d}^n x = (n-4)a_2(x). \tag{2.38}$$

Presumably a relation of this kind holds for all $a_k(x)$ for suitable ξ.

The structure of $\langle T_{\mu\nu}\rangle_{\rm div}$ can be made more transparent by expressing each of the tensors $H_{\mu\nu}$, ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$ in terms of the Weyl and Ricci tensors. The Weyl tensor in n dimensions is

$$C_{\alpha\beta\gamma\delta}=R_{\alpha\beta\gamma\delta}+(n-2)^{-1}(g_{\alpha\delta}R_{\beta\gamma}+g_{\beta\gamma}R_{\alpha\delta}-g_{\alpha\gamma}R_{\beta\delta}-g_{\beta\delta}R_{\alpha\gamma})$$
$$+(n-1)^{-1}(n-2)^{-1}(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma})R \tag{2.39}$$

hence

$$H_{\mu\nu}=\frac{1}{2}C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}g_{\mu\nu}-2C_{\mu\alpha\beta\gamma}C_{\nu}^{\alpha\beta\gamma}-\frac{4n}{n-2}C_{\alpha\mu\beta\nu}R^{\alpha\beta}+\frac{4(n^2-3n+4)}{(n-2)^2}R_{\mu\alpha}R_{\nu}^{\alpha}$$
$$-\frac{2n}{(n-2)^2}R^{\alpha\beta}R_{\alpha\beta}g_{\mu\nu}-\frac{4(n^2-2n+2)}{(n-1)(n-2)^2}RR_{\mu\nu}$$
$$+\frac{3n-2}{(n-1)(n-2)^2}R^2g_{\mu\nu}-4\Box R_{\mu\nu}+2R_{;\mu\nu} \tag{2.40}$$

$$^{(2)}H_{\mu\nu}=R_{,\mu\nu}-\Box R_{\mu\nu}-\tfrac{1}{2}\Box Rg_{\mu\nu}-2C_{\alpha\mu\beta\nu}R^{\alpha\beta}+\frac{4}{n-2}R_{\mu\alpha}R_{\nu}^{\alpha}$$
$$+\frac{n-6}{2(n-2)}R^{\alpha\beta}R_{\alpha\beta}g_{\mu\nu}-\frac{2n}{(n-1)(n-2)}RR_{\mu\nu}+\frac{2}{(n-1)(n-2)}R^2g_{\mu\nu}. \tag{2.41}$$

From equation (1.7) it follows that $H_{\mu\nu}-4^{(2)}H_{\mu\nu}+^{(1)}H_{\mu\nu}$ vanishes in four dimensions. In n dimensions,

$$H_{\mu\nu}-4^{(2)}H_{\mu\nu}+^{(1)}H_{\mu\nu}=\tfrac{1}{2}C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}g_{\mu\nu}-2C_{\mu\alpha\beta\gamma}C_{\nu}^{\alpha\beta\gamma}-(n-4)^{(3)}H_{\mu\nu} \tag{2.42}$$

where

$$^{(3)}H_{\mu\nu}=-\frac{4}{n-2}C_{\alpha\mu\beta\nu}R^{\alpha\beta}+\frac{2(n-3)}{(n-2)^2}R^{\alpha\beta}R_{\alpha\beta}g_{\mu\nu}-\frac{4(n-3)}{(n-2)^2}R_{\mu\alpha}R_{\nu}^{\alpha}$$
$$+\frac{2n(n-3)}{(n-1)(n-2)^2}RR_{\mu\nu}-\frac{(n+2)(n-3)}{2(n-1)(n-2)^2}R^2g_{\mu\nu}. \tag{2.43}$$

The tensor $^{(3)}H_{\mu\nu}$ is a generalisation of a tensor that has been encountered before in calculations of $\langle 0|T_{\mu\nu}|0\rangle_{\text{ren}}$ in conformally flat space-times (Davies *et al* 1977). When $C_{\alpha\mu\beta\nu}=0$ and $n=4$, equation (2.43) reduces to the first expression in equation (3.12) of Davies *et al* (1977) apart from differences in sign convention. However, the second expression in Davies *et al* (1977) (equation (3.12)) is not consistent with equation (2.43) when $C_{\alpha\mu\beta\nu}\neq 0$. To obtain a consistent expression, one would have to add $3C_{\alpha\mu\beta\nu}R^{\alpha\beta}$ to equation (2.43), but there is no particular reason to do this and equation (2.43) appears to be the more natural generalisation to non-conformally flat space-times. Substituting equations (2.42) and (2.43) in (2.30),

$$\langle T_{\mu\nu}\rangle_{\text{div}}=-\frac{1}{(4\pi)^{n/2}}\left[\frac{1}{n-4}+\frac{1}{2}(\gamma+\ln m^2/\kappa^2)\right]\left[\frac{m^4g_{\mu\nu}}{n(n-2)}+\frac{m^2(n-4)G_{\mu\nu}}{3(n-1)(n-2)}\right.$$
$$+\frac{1}{180}C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}g_{\mu\nu}-\frac{1}{45}C_{\mu\alpha\beta\gamma}C_{\nu}^{\alpha\beta\gamma}-\frac{n-4}{90}{}^{(3)}H_{\mu\nu}$$
$$\left.+\frac{1}{30}\left({}^{(2)}H_{\mu\nu}-\frac{1}{3}{}^{(1)}H_{\mu\nu}\right)\right] \tag{2.44}$$

where the substitution (2.2) has been made, and the last term in (2.30) dropped since it is of order $(n-4)$. Now write

$$ {}^{(2)}H_{\mu\nu} - \frac{1}{3}{}^{(1)}H_{\mu\nu} = \frac{(n-2)(n-4)}{4(n-3)}{}^{(3)}H_{\mu\nu} - \frac{n-4}{12(n-1)}{}^{(1)}H_{\mu\nu} + A_{\mu\nu} \qquad (2.45) $$

where

$$ A_{\mu\nu} \equiv -\frac{n}{4(n-1)}{}^{(1)}H_{\mu\nu} + {}^{(2)}H_{\mu\nu} - \frac{(n-2)(n-4)}{4(n-3)}{}^{(3)}H_{\mu\nu} \qquad (2.46) $$

$$ = \frac{n-2}{2(n-1)}R_{;\mu\nu} - \Box R_{\mu\nu} + \frac{1}{2(n-1)}\Box R g_{\mu\nu} - \frac{n-2}{n-3}C_{\alpha\mu\beta\nu}R^{\alpha\beta} + \frac{n}{n-2}R_{\mu\alpha}R^{\alpha}_{\nu} - \frac{1}{n-2}R^{\alpha\beta}R_{\alpha\beta}g_{\mu\nu} - \frac{n}{(n-1)(n-2)}RR_{\mu\nu} + \frac{1}{(n-1)(n-2)}R^2 g_{\mu\nu}. \qquad (2.47) $$

From equation (2.47) it follows that $A_{\mu\nu}$ is traceless in n dimensions, and from equation (2.45) one can see that, in four dimensions

$$ A_{\mu\nu} = {}^{(2)}H_{\mu\nu} - \tfrac{1}{3}{}^{(1)}H_{\mu\nu}. \qquad (2.48) $$

This vanishes in conformally flat space–times. Substituting equation (2.45) in equation (2.44),

$$ \langle T_{\mu\nu}\rangle_{\text{div}} = -\frac{1}{(4\pi)^{n/2}}\left[\frac{1}{n-4} + \frac{1}{2}(\gamma + \ln m^2/\kappa^2)\right] \times\left[\frac{4m^4 g_{\mu\nu}}{n(n-2)} + \frac{1}{180}C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}g_{\mu\nu} - \frac{1}{45}C_{\mu\alpha\beta\gamma}C_{\nu}^{\alpha\beta\gamma} + \frac{1}{30}A_{\mu\nu}\right] - \frac{1}{(4\pi)^{n/2}}\left[\frac{m^2 G_{\mu\nu}}{3(n-1)(n-2)} - \frac{1}{360(n-1)}{}^{(1)}H_{\mu\nu} - \frac{n-6}{360(n-3)}{}^{(3)}H_{\mu\nu}\right]. \qquad (2.49) $$

It is worth noting that since, in four dimensions (using equation (2.42) and the Gauss–Bonnet theorem)

$$ C_{\mu\alpha\beta\gamma}C_{\nu}^{\alpha\beta\gamma} = \tfrac{1}{4}C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}g_{\mu\nu} \qquad (2.50) $$

the coefficient of $\gamma + \ln(m^2/\kappa^2)$ may be taken to be

$$ \tfrac{1}{2}m^4 g_{\mu\nu} + \tfrac{1}{30}({}^{(2)}H_{\mu\nu} - \tfrac{1}{3}{}^{(1)}H_{\mu\nu}). \qquad (2.51) $$

Consequently, for a massless conformal field in a conformally invariant space–time there is no term involving $\ln(m^2/\kappa^2)$ and hence the renormalisation is unique. This is a reflection of the fact that in this case $\langle 0|T_{\mu\nu}|0\rangle_{\text{ren}}$ is determined by a choice of quantum state, $|0\rangle$, and the value of $\langle 0|T^{\alpha}_{\alpha}|0\rangle_{\text{ren}}$ which is independent of the renormalisation *ansatz* used (see remarks following equation (2.33)). If the state $|0\rangle$ is the vacuum state in the conformally related flat space–time, $\langle 0|T_{\mu\nu}|0\rangle_{\text{ren}}$ is given by (Davies *et al* 1977, Brown and Cassidy 1977)

$$ \langle 0|T_{\mu\nu}|0\rangle_{\text{ren}} = \frac{1}{2880\pi^2}[{}^{(3)}H_{\mu\nu} - \tfrac{1}{6}{}^{(1)}H_{\mu\nu}]. \qquad (2.52) $$

This result can be seen to arise from the subtraction of the finite term contained in the last bracket in equation (2.49), which for $m = 0$ and $n = 4$ is the negative of equation (2.52). Evidently the divergent quantity $\langle 0|T_{\mu\nu}|0\rangle$ is cancelled by the term proportional to $(n-4)^{-1}A_{\mu\nu}$, which is of course traceless.

The uniqueness of the renormalisation *ansatz* in this special case means that, for a given quantum state $|0\rangle$, there is no ambiguity in $\langle 0|T_{\mu\nu}|0\rangle_{\text{ren}}$. It does not mean, however, that if a different quantum state is chosen the renormalised stress tensor will be given by equation (2.52): it can differ from this by any traceless, conserved tensor (Bunch 1978).

The renormalised stress tensor in two dimensions can be obtained in a similar manner to that in four dimensions. It does not matter that there is no Einstein equation in two dimensions since renormalisation of the stress tensor takes place in n dimensions. Equation (2.23) is replaced by

$$\mathscr{L}^{(1)}(x) = -\frac{1}{(4\pi)^{n/2}}\left[\frac{1}{n-2}+\frac{1}{2}(\gamma+\ln m^2/\kappa^2)\right]\left[-\frac{2}{n}m^2a_0+a_1\right]$$
$$+\frac{1}{2(4\pi)^{n/2}}\sum_{k=2}^{\infty} a_k(x)m^{2-2k}\Gamma\left(k-\frac{n}{2}\right). \tag{2.53}$$

The regularisation parameter is now $\epsilon = n-2$. As before, there is no unique renormalisation *ansatz*. If all terms which are second order or lower in derivatives of the metric (*ansatz* (ii)) are discarded,

$$\mathscr{L}^{(1)}_{\text{div}}(x) = -\frac{1}{(4\pi)^{n/2}}\left[\frac{1}{n-2}+\frac{1}{2}(\gamma+\ln m^2/\kappa^2)\right]\left[-\frac{2m^2}{n}+\left(\frac{1}{6}-\xi\right)R\right]. \tag{2.54}$$

The divergences in the stress tensor are

$$\langle T_{\mu\nu}\rangle_{\text{div}} = -\frac{1}{(4\pi)^{n/2}}\left[\frac{1}{n-2}+\frac{1}{2}(\gamma+\ln m^2/\kappa^2)\right]\left[-\frac{2m^2}{n}g_{\mu\nu}+2\left(\xi-\frac{1}{6}\right)G_{\mu\nu}\right]. \tag{2.55}$$

These can be removed by renormalisation of the cosmological constant Λ and Newton's gravitational constant G. For a massless field, the trace of $\langle T_{\mu\nu}\rangle_{\text{div}}$ is finite and nonzero as $n \to 2$, since

$$g^{\mu\nu}G_{\mu\nu} = \left(1-\frac{n}{2}\right)R \tag{2.56}$$

and hence

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle_{\text{div}} = \frac{1}{(4\pi)^{n/2}}(\xi-\tfrac{1}{6})R. \tag{2.57}$$

Putting $n = 2$ requires $\xi = 0$ for conformal invariance, and so:

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle_{\text{div}} = -\frac{R}{24\pi}. \tag{2.58}$$

This is, as expected, the negative of the anomalous trace for a massless conformal scalar field in a two-dimensional space–time.

3. Discussion and comparison with some earlier work

In the previous section it was shown that by using dimensional regularisation it is possible to renormalise the quantum stress tensor in curved space–time by renormalising coupling constants in Einstein's equation, and that this renormalisation leads directly to the well known trace anomaly for conformally invariant fields. Because of the Gauss–Bonnet theorem, it appears that regularisation schemes which work entirely in two or four dimensions will not give rise to this kind of renormalisation. This shows up strikingly in two dimensions in which there is no Einstein equation at all, precisely because the two-dimensional Gauss–Bonnet theorem implies that the Einstein tensor vanishes identically. Consequently, dimensional regularisation appears to be necessary for the renormalisation of coupling constants in Einstein's equation. An attempt at such a renormalisation has been made before (Brown 1976, Brown and Cassidy 1976) but that work contains a number of important differences from the approach described in § 2. These differences will now be discussed in order to reduce any confusion that might otherwise arise. For simplicity a detailed comparison will only be made for the two-dimensional stress tensor, but the comments made below apply equally to the four-dimensional case.

In Brown (1976) it is shown that both the effective Lagrangian $\mathscr{L}^{(1)}(x)$ and the stress tensor $\langle T_{\mu\nu}\rangle$ can be expressed as integrals of the form:

$$I=\kappa^2\int_0^\infty \mathrm{i}\mathrm{d}s(\kappa^2\mathrm{i}s)^{-n/2}F(\mathrm{i}s;n). \tag{3.1}$$

Integrating by parts, assuming $n<2$ (continuation to other values of n can take place later):

$$I=-\frac{1}{1-n/2}\int_0^\infty(\kappa^2\mathrm{i}s)^{1-n/2}\mathrm{i}\mathrm{d}s\frac{\partial}{\partial \mathrm{i}s}F(\mathrm{i}s;n). \tag{3.2}$$

But

$$(\kappa^2\mathrm{i}s)^{1-n/2}\sim 1+(1-n/2)\ln(\kappa^2\mathrm{i}s)+O(n-2)^2 \tag{3.3}$$

and hence

$$I=\frac{1}{1-n/2}F(0;n)-\int_0^\infty \mathrm{i}\mathrm{d}s\ \ln(\kappa^2\mathrm{i}s)\frac{\partial}{\partial \mathrm{i}s}F(\mathrm{i}s;n). \tag{3.4}$$

For example, the effective Lagrangian may be expressed by this method as:

$$\mathscr{L}^{(1)}(x)=\frac{1}{1-n/2}\frac{1}{n(4\pi)^{n/2}}\frac{\partial}{\partial \mathrm{i}s}[\exp(-\mathrm{i}m^2s)F(x;\mathrm{i}s)]_{s=0}$$
$$-\frac{1}{n(4\pi)^{n/2}}\int_0^\infty \mathrm{i}\mathrm{d}s\ \ln(\kappa^2\mathrm{i}s)\left(\frac{\partial}{\partial \mathrm{i}s}\right)^2[\exp(-\mathrm{i}m^2s)F(x;\mathrm{i}s)]. \tag{3.5}$$

The functions $F(\mathrm{i}s;n)$ and $F(x;\mathrm{i}s)$ appearing in equations (3.1) and (3.5) are, of course, not the same. One can check that substituting equation (2.13) in equation (3.5) gives rise to equation (2.53). In equation (3.5), the first term is the pole term; the $\ln(m^2/\kappa^2)$ term is contained in the second term in equation (3.5). Discarding the entire first term would then be equivalent to *ansatz* (i). However, in Brown (1976) an attempt is made to continue back to two dimensions before the pole term is discarded. For example, this means that the first term in equation (3.4) is written as

$$\frac{1}{1-n/2}F(0;n)=\frac{1}{1-n/2}F(0;2)-2\frac{\partial F}{\partial n}\bigg|_{n=2} \tag{3.6}$$

and the term $(1-n/2)^{-1}F(O;2)$ is considered to be the pole term which is to be discarded. Brown also discards some of the finite term in equation (3.6), that arising from the explicit n dependence of $(4\pi)^{n/2}$. The divergent part of the effective action is then

$$W_{\text{div}}=\frac{1}{4\pi}\int \sqrt{g}\left(\frac{1}{2-n}+L_2\right)(\tfrac{1}{6}R-m^2)\,\mathrm{d}^2x \tag{3.7}$$

where $L_2=\frac{1}{2}\ln 4\pi$. Because the continuation to two dimensions has already taken place, the term $\int \sqrt{g}R\ \mathrm{d}^2x$, which is a metric-independent constant, can be omitted from equation (3.7) leaving

$$W_{\text{div}}=-\frac{m^2}{4\pi}\left(\frac{1}{2-n}+L_2\right)\int \sqrt{g}\,\mathrm{d}^2x. \tag{3.8}$$

The corresponding divergences in the stress tensor are shown in Brown (1976) to be

$$\langle T_{\mu\nu}\rangle_{\text{div}}=-\frac{m^2}{4\pi}\left(\frac{1}{2-n}+L_2\right)g_{\mu\nu}. \tag{3.9}$$

These are indeed the variational derivative of equation (3.8). However, it is quite clear that subtracting equation (3.9) from $\langle 0|T_{\mu\nu}|0\rangle$ will not give rise to an anomalous trace when $m=0$. A similar procedure is used in the four-dimensional calculation and the divergences in the stress tensor are again traceless when $m=0$ (see equations (1.35)–(1.40) of Brown 1976). How then does Brown obtain an anomalous trace? The answer is that a nonzero trace is apparently obtained in those terms in $\langle T_{\mu\nu}\rangle$ which remain after the pole terms (3.9) have been removed. Brown calls these terms $\langle T_{\mu\nu}\rangle_{\text{ren}}$, but it should be remembered that the renormalised quantity which acts as the source in Einstein's equations is defined by equation (1.2). The trace of $\langle T_{\mu\nu}\rangle_{\text{div}}$ should be the negative of the trace anomaly, and since Brown has in fact subtracted only a term which is traceless when $m=0$, what remains should have a trace which is the negative of that claimed. To clear up the confusion arising from this, consider equation (3.20) of Brown (1976):

$$\langle T_{\mu\nu}\rangle=\frac{1}{(4\pi)^{n/2}}\int_0^\infty \frac{\mathrm{i}\mathrm{d}s}{(\mathrm{i}s)^{n/2}}\exp(-\mathrm{i}m^2s)T_{\mu\nu}(x;\mathrm{i}s;n) \tag{3.10}$$

where $T_{\mu\nu}(x;\mathrm{i}s;n)$ is given by equation (3.21) of Brown (1976). This is of the form (3.1) and hence may be expressed according to equation (3.4) as

$$\langle T_{\mu\nu}\rangle=-\frac{2}{(4\pi)^{n/2}(n-2)}T_{\mu\nu}(x;0;n)$$
$$-\frac{1}{(4\pi)^{n/2}}\int_0 \mathrm{i}\mathrm{d}s\,\ln(\kappa^2\mathrm{i}s)\frac{\partial}{\partial \mathrm{i}s}(\exp(-\mathrm{i}m^2s)T_{\mu\nu}(x;\mathrm{i}s;n)). \tag{3.11}$$

Brown now expands the first term about $n=2$, discards the pole term and claims that when $m=0$ the trace of the remainder is $R/24\pi$. However, using equation (3.21) of Brown (1976) it can be seen that the first term in equation (3.11) above is

$$\frac{2m^2 g_{\mu\nu}}{(4\pi)^{n/2} n(n-2)} + \frac{a_1(x)}{(4\pi)^{n/2}} \frac{g_{\mu\nu}}{n} + \frac{2}{(4\pi)^{n/2}(n-2)}(\tfrac{1}{6}-\xi)G_{\mu\nu}. \tag{3.12}$$

The first term in equation (3.12) is the term which Brown discards. The second term has trace $R/24\pi$. He also discards the third term on the grounds that $G_{\mu\nu}$ vanishes in two dimensions. But clearly

$$\lim_{n\to 2} \frac{G_{\mu\nu}}{n-2} \neq 0 \tag{3.13}$$

since it has nonzero trace. Indeed, the trace of the third term in (3.12) is $-R/24\pi$, so that equation (3.12) is actually traceless when $m=0$. There are also terms of second order in the derivatives of the metric which have nonzero trace when $m=0$ contained in the second term of equation (3.11). Writing this term as

$$-\frac{1}{(4\pi)^{n/2}} \int_0^\infty \mathrm{i}\,\mathrm{d}s \ln(\kappa^2 \mathrm{i}s) \exp(-\mathrm{i}m^2 s)\left(-m^2 T_{\mu\nu} + \frac{\partial}{\partial \mathrm{i}s} T_{\mu\nu}\right) \tag{3.14}$$

and using equation (3.21) of Brown (1976) and the expansion (2.13) one obtains four terms which are of second order in derivatives of the metric, the first three coming from $-m^2 T_{\mu\nu}$ and the fourth from $\partial T_{\mu\nu}/\partial \mathrm{i}s$:

$$-\frac{1}{(4\pi)^{n/2}} \int_0^\infty \mathrm{i}\,\mathrm{d}s \ln(\kappa^2 \mathrm{i}s) \exp(-\mathrm{i}m^2 s)\left[\frac{m^4}{n} g_{\mu\nu} a_1(x) \mathrm{i}s - \frac{m^2(2-n)}{2n} g_{\mu\nu} a_1(x) \right.$$

$$\left. + m^2(\tfrac{1}{6}-\xi) G_{\mu\nu} - \frac{m^2}{n} g_{\mu\nu} a_1(x)\right] \tag{3.15}$$

$$= -\frac{m^2}{(4\pi)^{n/2}} \int_0^\infty \mathrm{i}\,\mathrm{d}s \ln(\kappa^2 \mathrm{i}s) \exp(-\mathrm{i}m^2 s)$$

$$\times \left[g_{\mu\nu} a_1(x) \left(\frac{m^2 \mathrm{i}s}{n} + \frac{n-4}{2n}\right) + (\tfrac{1}{6}-\xi) G_{\mu\nu}\right]. \tag{3.16}$$

This integral can be evaluated using expression (4.352) of Gradshteyn and Ryzhik (1965) to give

$$\frac{\xi - \frac{1}{6}}{(4\pi)^{n/2}} \left[R g_{\mu\nu} \left(\frac{1}{n} + \frac{2-n}{2n}(\gamma + \ln m^2/\kappa^2)\right) - G_{\mu\nu}(\gamma + \ln m^2/\kappa^2)\right]. \tag{3.17}$$

The trace of equation (3.17) is, for arbitrary n,

$$\frac{\xi - \frac{1}{6}}{(4\pi)^{n/2}} R. \tag{3.18}$$

Putting $n=2$, $\xi=0$ gives $-R/24\pi$. Thus the terms in equation (3.11) which Brown fails to discard do indeed have a trace equal to $-R/24\pi$, the negative of the trace anomaly, and not $R/24\pi$ as Brown himself claims. From equations (3.12) and (3.17) it can be seen that this is true whether one intends to renormalise using *ansatz* (i) or *ansatz* (ii). The pole terms alone (the first and third terms in equation (3.12)) have a trace $-R/24\pi$ when $m=0$, as does the entire term of second order in derivatives of the metric (the sum of equations (3.12) and (3.17)).

Finally, it should be emphasised that the problem of how to obtain $\langle 0|T_{\mu\nu}|0\rangle_{\rm ren}$ in a particular space–time with a particular quantum state $|0\rangle$ is not yet complete. This paper has described how $\langle 0|T_{\mu\nu}|0\rangle_{\rm div}$ can be calculated in an arbitrary space–time and how it can be removed by renormalisation, but it has not even started to consider how to obtain the dimensionally regularised quantity $\langle 0|T_{\mu\nu}|0\rangle$ which, according to equation (1.2), must be known before $\langle 0|T_{\mu\nu}|0\rangle_{\rm ren}$ can be calculated. In this respect, dimensional regularisation at its present stage of development is a less practical regularisation scheme than covariant point-splitting which has been used to calculate $\langle 0|T_{\mu\nu}|0\rangle_{\rm ren}$ for a number of different space–times (Davies *et al* 1976, Christensen 1976, Davies *et al* 1977, Bunch and Davies 1978a, b), Bunch (1977, 1978)). However, if this difficulty could be overcome, dimensional regularisation might prove to be a more efficient means of obtaining $\langle 0|T_{\mu\nu}|0\rangle_{\rm ren}$.

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